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A Mixture Model Based on Birnbaum-Saunders Distributions

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Abstract

In this paper, we introduce a new mixture models based on Birnbaum-Saunders distributions, which is highly flexible and contains several well known probability distributions. We derive the density and distribution functions and moments as well as some properties and transformations of the new model. Finally, we apply the results to real data sets using maximum likelihood methods. The illustration shows the adequacy of the new model.

Keywords Mixture distributions; Length-biased distributions; Moments; Skewness.

1 Introduction

Mixture distributions play an important role in statistics. Particularly, finite mixture models have been used to solve diverse theoretical and practical applications. The theoretical applications include Bayesian analysis, diagnostics techniques, nonparametric statistics, and statistical modeling, among others. The practical applications appear, for instance, in biological sciences, economics, fisheries research, lifetime studies, and sociological surveys. These applications arise because frequently data can be considered as coming from a population where the random variable (r.v.) of interest follows a mixture of two or more distributions. This leads us to think about mixing probability

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models producing new distributions. Mixture distributions have been considered by different authors; for a review of estimation methods and applications, see Everitt and Hand (1981), Titterington et al. (1985), Lindsay (1995), Maclachlan and Peel (2000), and Böhning et al. (2007). In addition, Al-Hussaini and Sultan (2001) reviewed some characteristics of finite mixtures of certain lifetime models.

A model that has received great attention in the last time is the Birnbaum-Saunders (BS) distribution; see Sanhueza et al. (2008) for a complete review about this distribution and its applications. This model is an important positively skewed distribution and has attractive properties. Due to the interesting theoretical arguments established in its genesis, the BS distribution have been transformed in a model largely applied to study lifetimes. However, the BS model has been used beyond the lifetime analysis. For more details about of the BS model, see Johnson et al. (1995).

On the other hand, the length-biased (LB) distributions are particular cases of the weighted distributions, which arise from an interesting scope about sampling procedure; see Patil (2002). The LB distributions have been applied in diverse subjects, such as biometry, ecology, environmental, and lifetime analysis. A review of these distributions and their applications is included in Gupta and Kirmani (1990). Length-biased versions for several distributions have been developed. For example, the LB versions of the inverse Gaussian (IG) and lognormal distributions can be revised in Sansgiry and Akman (2001). An interesting mixture model based on the IG distribution were proposed by Jörgensen et al. (1991), Gupta and Akman (1995), and Kotz et al. (1991).

In this article, we develop a new mixture model based on the BS distribution. This model is called the mixture Birnbaum-Saunders (MBS) distribution. The new distribution is widely flexible and its applicability is seen with great perspective. In the rest of the paper, we give a background of the BS and LB distributions. Next, we provide a comprehensive description of the statistical and mathematical properties of the new mixture distribution. Among the mathematical properties, we consider the probability density function (pdf), cumulative distribution function (cdf), survival function (s.f.), hazard function (h.f.), and moments. For statistical aspects, we consider maximum likelihood (ML) estimation. Finally, some of the obtained results are considered to illustrate the applicability of the new model.

2 Background

In this section, we present a background related to BS and LB distributions.

2.1 The Birnbaum-Saunders distribution

When an r.v. T follows the BS distribution with shape and scale parameters, α and β, respectively, then the notation $T \sim \mathrm{BS}(\alpha, \beta)$ is used. Thus, if

$$Z = \frac{1}{\alpha}\left[\sqrt{\frac{T}{\beta}} - \sqrt{\frac{\beta}{T}}\right] \sim \mathrm{N}(0, 1), \tag{1}$$

then

$$T = \frac{\beta}{4}\left[\alpha Z + \sqrt{\{\alpha Z\}^2 + 4}\right]^2 \sim \mathrm{BS}(\alpha, \beta), \tag{2}$$

where $\alpha > 0$ and $\beta > 0$. The result given in Eq. (1) leads to

$$W = Z^2 = \frac{1}{\alpha^2}\left[\frac{T}{\beta} + \frac{\beta}{T} - 2\right] \sim \chi^2(1).$$

The pdf and cdf of T are, respectively, given by

$$f_T(t) = \phi(a_t)\, A_t \quad \text{and} \quad F_T(t) = \Phi(a_t), \quad t > 0, \tag{3}$$

where

$$a_t = a_t(\alpha, \beta) = \frac{1}{\alpha}\left[\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right], \quad A_t = \frac{d}{dt}a_t = \frac{1}{2\alpha\beta}\left[\sqrt{\frac{\beta}{t}} + \sqrt{\frac{\beta}{t}}\right],$$

and $\phi(\cdot)$ and $\Phi(\cdot)$ denote the standard normal pdf and cdf, respectively. Some properties of T are: $cT \sim \mathrm{BS}(\alpha, c\beta)$, with $c > 0$, and $1/T \sim \mathrm{BS}\,(\alpha, 1/\beta)$. The quantile function of T is given by

$$t(q) = \frac{\beta}{4}\left[\alpha\, z(q) + \sqrt{\{\alpha z(p)\}^2 + 4}\right]^2,$$

with $0 < q < 1$, where $z(p)$ is the qth $\mathrm{N}(0, 1)$ quantile. Thus, since $t(0.5) = \beta$, then β is the median. The rth moment of T is

$$\mathbb{E}\left[T^r\right] = \beta^r \sum_{j=0}^{r} \binom{2r}{2j} \sum_{i=0}^{j} \binom{j}{i} \frac{(2r - 2j + 2i)!}{2^{r-j+i}(r - j + i)!} \left[\frac{\alpha}{2}\right]^{2r-2j+2i}. \tag{4}$$

Based on Eq. (4), the mean and the variance are

$$\mathbb{E}[T] = \frac{\beta}{2}[\alpha^2 + 2] \quad \text{and} \quad \mathrm{Var}[T] = \frac{\beta^2}{4}[5\alpha^4 + 4\alpha^2],$$

while the coefficients of variation (CV), skewness (CS), and kurtosis (CK) are given by

$$\gamma = \frac{\sqrt{5\alpha^4 + 4\alpha^2}}{\alpha^2 + 2}, \quad \delta = \frac{44\alpha^3 + 24\alpha}{\left[\sqrt{5\alpha^2 + 4}\right]^3}, \quad \text{and} \quad \kappa = 3 + \frac{558\alpha^4 + 240\alpha^2}{[5\alpha^2 + 4]^2},$$

respectively. On the other hand, the BS distribution degenerate at the β parameter; see Kundu et al. (2008). The coefficients γ, δ, and κ are invariant under scale, i.e., these indicators are independent of the parameter β. The skewness and kurtosis are controlled by the parameter α.

2.2 Length-biased distributions

As mentioned, the LB distributions are particular cases of the weighted distributions. Specifically, if Y is a nonnegative r.v. with pdf $f_Y(y)$, then the weighted version of Y with weight function $w(y)$, which we denote by the r.v. Y_w and whose distribution is called the weighted distribution, has pdf given by

$$f_{Y_w}(y) = \frac{w(y) f_Y(y)}{\mathbb{E}[w(Y)]}; \quad y > 0, \tag{5}$$

assuming that $\mathbb{E}[w(Y)] < \infty$, i.e., the first moment of $w(Y)$ exists. A particular case of the weighted distributions is obtained when we replace $w(y) = y$ in Eq. (5). In this case, Y_w is called the size-biased or length-biased version of Y, denoted by the r.v. L, which has pdf expressed as

$$f_L(l) = \frac{l\, f_Y(l)}{\mu}, \quad l > 0, \mu > 0, \tag{6}$$

where $\mu = \mathbb{E}(Y) < \infty$. The distribution of L is the LB version of the distribution of Y. The distribution of L is the LB version of the distribution of Y. The parameter that corresponds to the mean from the original distribution is incorporated in the length-biased density with the goal of obtaining a valid pdf. Actually, the mean is a function of the parameters from the original model. Thus, the LB distribution does not incorporate new parameters and has the same amount of parameters as the original model. Thus, if $T \sim \text{BS}(\alpha, \beta)$, then the length-biased version of T has pdf given by

$$f_L(l) = \phi(a_l)\, \frac{1}{[\alpha^3 + 2\alpha]\, \beta} \left[\sqrt{\frac{l}{\beta}} + \sqrt{\frac{\beta}{l}}\right], \quad l > 0,\ \alpha > 0,\ \beta > 0, \tag{7}$$

where a_l is defined below Eq. (3). The distribution of L with pdf given in Eq. (7) is called the length-biased Birnbaum-Saunders (LBS) distribution. For more details about the LBS distribution, see Leiva et al. (2009).

3 The mixture Birnbaum-Saunders distribution

In this section, we develop the MBS distribution. For this new distribution, we present the pdf, cdf, h.f., s.f., moments, and discuss some properties.

3.1 Shape analysis

The mixture of the BS and LBS distributions has its pdf expressed in a usual notation as

$$f_X(x) = [1-p]\, f_T(x) + p\, f_L(x), \quad x > 0, \tag{8}$$

where the densities $f_T(\cdot)$ and $f_L(\cdot)$ are defined by Eqs. (3) and (7), respectively. Thus, explicitly, the pdf of the r.v. X given in Eq. (8) is

$$f_X(x) = \phi(a_x)\left[(1-p)A_x + p\frac{b_x}{(\alpha^2+2)\beta}\right], \quad x > 0,\ \alpha > 0,\ \beta > 0, 0 < p < 1, \tag{9}$$

which is denoted by $X \sim \mathrm{MBS}(\alpha, \beta, p)$, with p being a mixing parameter, and where

$$b_x = \frac{1}{\alpha}\left[\sqrt{\frac{x}{\beta}} + \sqrt{\frac{\beta}{x}}\right]$$

and a_x and A_x are given analogously to the ones presented in Eq. (3).

Standard calculations based on the derivative of $f_X(x)$ allow for finding the mode of X, denoted by x_m, which is obtained by the solution of the equation

$$[1-p]\left[a_{x_m}\, A^2_{x_m} - A'_{x_m}\right] + p\frac{1}{[\alpha^2+2]\,\beta}\left[a_{x_m}\, b_{x_m}\, A_{x_m} - B_{x_m}\right] = 0, \tag{10}$$

where

$$A'_x = \frac{d}{dx}A_x = -\frac{1}{4\alpha\beta x}\left[\sqrt{\frac{\beta}{x}} + 3\sqrt{\frac{\beta^3}{x^3}}\right], \quad B_x = \frac{d}{dx}b_x = \frac{1}{2\alpha\beta}\left[\sqrt{\frac{\beta}{x}} - \sqrt{\frac{\beta^3}{x^3}}\right],$$

and b_x is given as in Eq. (9). Fig. 1 shows density plots for different values of α, β, and p. We note that β only modifies the scale. However, α modifies the asymmetry and kurtosis. The mixing parameter p modifies the scale and kurtosis.

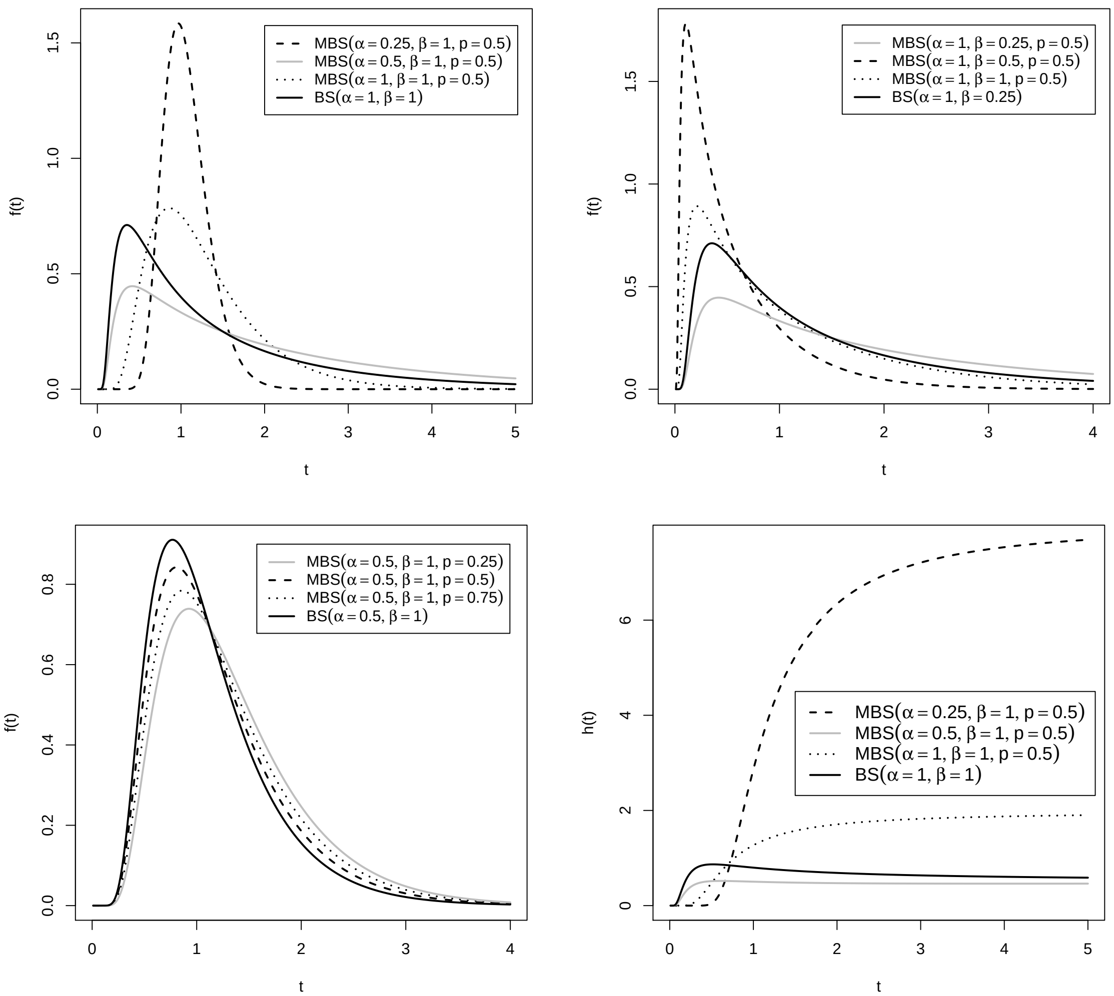

Figure 1: pdf and h.f. graphs of MBS distributions for the indicated values.

3.2 Properties

Let $X \sim \mathrm{MBS}(\alpha, \beta, p)$. Then, the following properties hold:

(P1) $c\,X \sim \mathrm{LBS}(\alpha, c\,\beta, p)$, with $c > 0$.

(P2) The r.v.

$$U = \frac{1}{\alpha^2}\left[\frac{X}{\beta} + \frac{\beta}{X} - 2\right]$$

is a mixture of two gamma distributions with mixing parameter w, i.e. the pdf of the r.v. U can be expressed by

$$f_U(u) = [1 - w] f_{U_1}(u) + w f_{U_2}(u),$$

where $w = p\,\alpha^2/[\alpha^2+2]$, $U_1 \sim \text{Gamma}(1/2,2)$, with $\mathbb{E}[U_1]=1$, and $U_2 \sim \text{Gamma}(3/2,2)$, with $\mathbb{E}[U_2]=3$. Therefore, the expected value of U is $\mathbb{E}[U]=4$.

(P3) The cdf of X is given by

$$F_X(x) = \Phi(a_x) + p\frac{\alpha^2}{[2+\alpha^2]}\left[\exp\left(\frac{2}{\alpha^2}\right)\left\{\Phi\left(\frac{\sqrt{4+\alpha^2 a_x^2}}{\alpha}\right)-1\right\} - \phi(a_x)\left\{a_x + \frac{\sqrt{4+\alpha^2 a_x^2}}{\alpha}\right\}\right], \quad x>0.$$

3.3 Survival and hazard functions

The s.f. and h.f. are, respectively, defined in general by $S_X(x)=1-F_X(x)$ and $h_X(x)=f_X(x)/S_X(x)$, with $x>0$, $f_X(x)>0$, and $0<S_X(x)<1$, where $f_X(x)$ and $S_X(x)$ are the pdf and s.f. of an r.v. X, respectively. Let $X \sim \text{MBS}(\alpha,\beta,p)$. Then, the s.f. and the h.f. of X are, respectively, expressed as

$$S_X(x) = \Phi(-a_x) - \frac{p\,\alpha^2}{[2+\alpha^2]}\left[\exp\left(\frac{2}{\alpha^2}\right)\left\{\Phi\left(\frac{\sqrt{4+\alpha^2 a_x^2}}{\alpha}\right)-1\right\} - \phi(a_x)\left\{a_x + \frac{\sqrt{4+\alpha^2 a_x^2}}{\alpha}\right\}\right]. \tag{11}$$

and

$$h_X(x) = \frac{\phi(a_x)\,\left[(1-p)A_x + p\frac{b_x}{(\alpha^2+2)\beta}\right]}{\Phi(-a_x) + \frac{p\,\alpha^2}{[2+\alpha^2]}\left[\phi(a_x)\left\{a_x + \frac{\sqrt{4+\alpha^2 a_x^2}}{\alpha}\right\} - \exp\left(\frac{2}{\alpha^2}\right)\left\{\Phi\left(\frac{\sqrt{4+\alpha^2 a_x^2}}{\alpha}\right)-1\right\}\right]}. \tag{12}$$

Fig. 1 shows plots of hazard functions for different values of α.

3.4 TTT curve

It is possible to characterize the h.f. of an r.v. X by using its corresponding total time on test (TTT) or its scaled TTT, which are

$$H_X^{-1}(u) = \int_0^{F_X^{-1}(u)} S_X(y)\,dy \quad \text{and} \quad W_X(u) = \frac{H_X^{-1}(u)}{H_X^{-1}(1)}, \quad 0<u<1, \tag{13}$$

respectively, where $F_X^{-1}(\cdot)$ is the inverse cdf of X and $S_X(\cdot)$ its SF. Now, $W_X(\cdot)$ can be empirically approximated by

$$W_n\left(\tfrac{k}{n}\right) = \sum_{i=1}^{k} X_{(i)} + [n-k]\, X_{(k)}, \quad k = 1, \ldots, n, \tag{14}$$

where $X_{(i)}$ is the corresponding ith order statistic. Thus, by means of the empirical scaled TTT curve given by

$$\left(\tfrac{k}{n}, W_n\left(\tfrac{k}{n}\right)\right), \quad k = 1, \ldots, n, \tag{15}$$

we can detect the type of HR that the data have. For example, from Fig. 2, it is possible to observe different theoretical shapes for the scaled TTT curve, which are associated with their respective HR; for more details about the TTT method, see Hoyland and Rausand (1994, pp. 493-495). Specifically, if the TTT curve is concave (or convex), then the IFR (or DFR) class is appropriate. Now, if the TTT curve is first concave (or convex) and then convex (or concave), a $\cap$(or $\cup$)-shaped HR must be considered. Of course, if the TTT curve is a straight line, the exponential distribution must be employed. Thus, a graphical plot of the empirical scaled TTT curve can give us an indication of the type of distribution associated with environmental data.

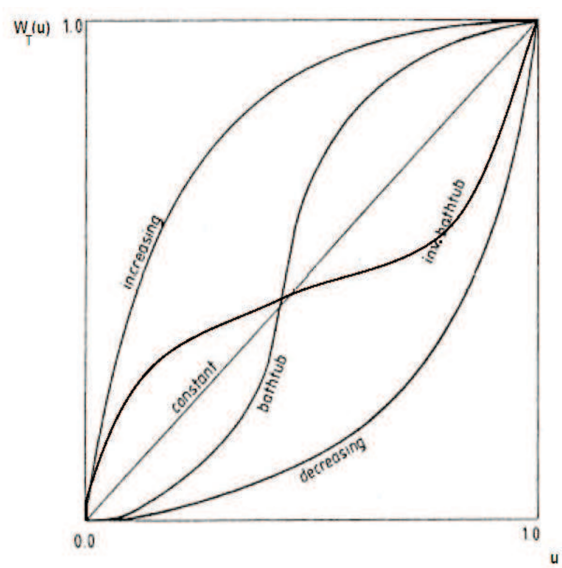

Figure 2: Scaled TTT curves for distributions with the indicated HR shape.

3.5 Moments

The moments of the MBS distribution depend on the moments of the BS distribution. Specifically, if $X \sim \text{MBS}(\alpha, \beta, p)$, then the rth moment of X is given by

$$\mathbb{E}\left[X^r\right] = [1-p]\mathbb{E}\left[T^r\right] + p\frac{2}{\beta[2+\alpha^2]}\mathbb{E}\left[T^{r+1}\right], \tag{16}$$

where $T \sim \text{BS}(\alpha, \beta)$ and

$$\mathbb{E}\left[T^r\right] = \frac{2^{r-1}\Gamma\left(r+\frac{1}{2}\right)\alpha^{2r}\beta^{2r}}{\sqrt{\pi}} - \frac{1}{2}\left\{\sum_{k=1}^{r}\binom{2r}{k}\frac{\mathbb{E}\left[T^{r-k}\right]}{\beta^{-k}[-1]^k} - \sum_{k=r+1}^{2r-1}\binom{2r}{k}\frac{\mathbb{E}\left[T^{k-r}\right]}{\beta^{k}[-1]^k}\right\}. \tag{17}$$

The first four moments of the MBS distribution can be obtained directly from Eqs. (16) and (17). Then, it is possible to compute the mean, variance, CV, CS, and CK of the MBS model.

3.6 Estimation

In this section, we estimate the parameters of the MBS model based on ML methods. The log-likelihood function for a random sample $\boldsymbol{T} = [T_1, \ldots, T_n]^\top$ of the model given in Eq. (9) is $l(\boldsymbol{\theta}) = \sum_{i=1}^{n} l_i(\boldsymbol{\theta})$, where $\boldsymbol{\theta} = [\alpha, \beta, p]^\top$ and

$$\sum_{i=1}^{n} l_i(\boldsymbol{\theta}) \propto -\frac{1}{2}\sum_{i=1}^{n} a_{t_i}^2 + \sum_{i=1}^{n}\log\left([1-p]A_{t_i} + \frac{p}{[\alpha^2+2]\beta}b_{t_i}\right). \tag{18}$$

The ML estimator of the parameter $\boldsymbol{\theta}$ is the solution of the equation $\dot{\mathbf{L}} = \mathbf{0}$, where $\dot{\mathbf{L}} = [\dot{\text{L}}_\alpha, \dot{\text{L}}_\beta, \dot{\text{L}}_p]^\top$ is the vector with first derivatives given by

$$\dot{\text{L}}_\alpha = \frac{1}{\alpha}\sum_{i=1}^{n} a_{t_i}^2 - \frac{1}{\alpha}\sum_{i=1}^{n}\frac{[1-p]\,[\alpha^2+2]\beta A_{t_i} + p\,b_{t_i} + 2p\frac{\alpha^2}{\alpha^2+2}b_{t_i}}{[1-p][\alpha^2+2]\beta A_{t_i} + p\,b_{t_i}},$$

$$\dot{\text{L}}_\beta = \frac{1}{2\beta}\sum_{i=1}^{n} a_{t_i}b_{t_i} + \frac{1}{2\beta}\sum_{i=1}^{n}\frac{[1-p][\alpha^2+2]\left[\beta A_{t_i} + \frac{\beta^{3/2}}{\alpha t_i^{3/2}}\right] - p\,a_{t_i} + 2b_{t_i}}{[1-p][\alpha^2+2]\beta A_{t_i} + p\,b_{t_i}}, \tag{19}$$

and

$$\dot{\text{L}}_p = \sum_{i=1}^{n}\frac{b_{t_i} - [\alpha^2+2]\beta A_{t_i}}{[1-p]A_{t_i}[\alpha^2+2]\beta + p\,b_{t_i}}.$$

However, the likelihood equations do not present analytical solutions so that the use of numerical iterative methods is necessary. As starting values can be considered the ML estimates of α and β of the BS distribution and $p = 1/2$.

4 Illustrative example

In this section, for the purpose of illustration, we analyze two data sets, which come from: (i) behavior sciences and (ii) engineering. The first case involves a new data set that have not been analyzed until now. The descriptions and data (with frequency in parentheses and none in parentheses when the frequency is one) are presented next. Using these data sets, we first carry out an exploratory data analysis (EDA) and then, by means of goodness-of-fit tools, we check the fitting of the model to the data sets. In the analyzes presented here, the parameters of the MBS model were estimated by using ML methods. These analyses are performed through an `R` package (`http://www.R-project.org`) developed by the authors called `mbs`, which is available upon request. The `mbs` package contains probabilistic, statistical, and model checking tools. Specifically, this package has incorporated the pdf, cdf, q.f., h.f., s.f., a random number generator, and ML estimation for the MBS distribution.

4.1 Data sets

Depressive condition data (S1). The scale "general rating of affective symptoms for preschoolers" (GRASP) measures behavioral and emotional problems of children, who can be classified with depressive condition or not according to this scale. A study conducted by the authors in a city located at the south part of Chile has allowed to collect real data (unpublished) corresponding to the scores of the GRASP scale of children, which are: 19(16), 20(15), 21(14), 22(9), 23(12), 24(10), 25(6), 26(9), 27(8), 28(5), 29(6), 30(4), 31(3), 32(4), 33, 34, 35(4), 36(2), 37(2), 39, 42, 44.

Fatigue life data (S2). Birnbaum and Saunders (1969b) reported data corresponding to cycles of aluminum specimens of type 6061-T6 until the failure. They were exposed to a pressure with maximum stress of 31,000 psi. The data ($\times 10^{-3}$) are the following: 70, 90, 96, 97, 99, 100, 103, 104(2), 105, 107, 108(3), 109(2), 112(2), 113, 114(3), 116, 119, 120(3), 121(2), 123, 124(5), 128(2), 129(2), 130(3), 131(5), 132(3), 133, 134(5), 136(2), 137, 138(3), 139(2), 141(2), 142(6), 144(2), 145, 146, 148(2), 149, 151(2), 152, 155, 156, 157(4), 158, 159, 162, 163(2), 164, 166(2), 168, 170, 174, 196, 212.

4.2 Exploratory data analysis

The data sets (S1 and S2) consist of 134 and 101 units of observations, respectively, with values of GRASP scale and lifetimes varying between 19 and 44 points, and 70 and 212 psi ($\times 10^{-3}$), respectively. The sample median and mean are: 24 and 25 points for S1, and 133.73 and 133.00 psi ($\times 10^{-3}$) for S2. The sample standard deviation, CV, CS, and CK are: 5.36 point, 21.47%, 1.11, and 0.89 for S1, and 22.36 psi ($\times 10^{-3}$), 16.70%, 0.33, and 0.97 for S2. Figure 4 displays histograms of S1 and S2. This EDA suggests positively skewed distributions with moderate degrees of variability, skewness, and kurtosis. MBS models seem to account quite well for the degrees of skewness and kurtosis presented in both data sets. In addition, scaled TTT curves corresponding to both data sets are shown in Fig. 3. This figure indicates that distributions with $\cap$-shaped hazard rates seem to be appropriate for modeling both data sets, so that the BS and MBS distributions could be good candidates for modeling these data sets; see theoretical plot in Fig. 2.

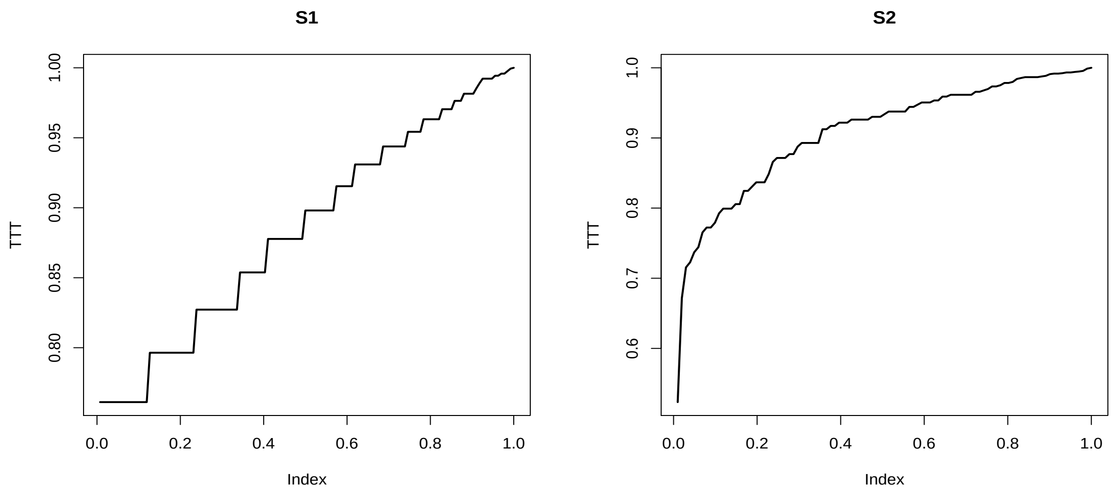

Figure 3: Scaled TTT curves for S1 and S2.

4.3 Implementation, estimation, and checking model

Several `R` packages are available from CRAN (`http://CRAN.R-project.org`) for analyzing data from different distributions. In particular, the `bs` and `ig` packages can be considered for the BS and IG distributions, respectively; see Leiva et al. (2006, 2008). We have developed a new `R` package named `mbs` to analyze data from the MBS distribution, which is available upon request. This package contains diverse probabilistic and lifetime indicators including the pdf, cdf, q.f., h.r., and s.f. of the MBS model. In addition, the `mbs` package has incorporated the scaled TTT curve as descriptive tool to identify the possible shape of the HR. On the other hand, the `mbs` package allows us to compute ML estimates of the parameters of the MBS distribution through the function `mlembs()`. We have used an `R` function called `nlm()` to compute the ML estimates of the parameters α, β and p of the MBS distribution. The command `nlm()` minimizes

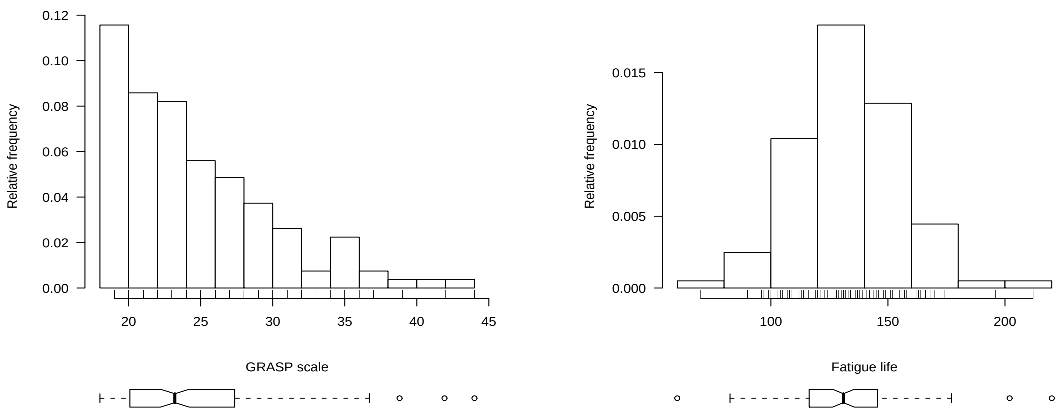

Figure 4: histograms and boxplots of S1 (left) and S2 (right).

the negative value of the likelihood function using a Newton-type algorithm. By using the `bs` package, we calculate the ML estimates of α and β of the BS distribution, which we employ as starting values for the numerical iterative procedure. Thus, based on the data sets upon analysis and the `mbs` package, we obtain the ML estimates of α, β, and p given by: $\hat{\alpha} = 0.171$ and $\hat{\beta} = 129.915$, and $\hat{p} = 0.01$ for S1, and $\hat{\alpha} = 0.201$ and $\hat{\beta} = 23.992$, and $\hat{p} = 0.00$ for S2.

Concluding Remarks

This article has dealt with a new mixture model based on Birnbaum-Saunders distributions. The MBS model is more flexible than the corresponding classical BS one for modeling different types of data as those used in the application part. A characterization of the new distribution and a hazard analysis have been carried out. This analysis has shown that it is possible to model the hazard in a flexible way allowing different forms with the MBS distribution. Specifically, we have found the pdf of the MBS model and studied its shape. We have computed the cdf and moments and highlighted properties and transformations related to the proposed model. We have implemented an `R` package for the new distribution, which makes this model more attractive to users in several disciplines. Finally, we have illustrated the obtained results by using two real data sets. This illustration has shown the adequacy and flexibility of the new model to the considered data sets.

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